Formulas for Some Restricted Partition Functions

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Abstract

The purposes of this research are to find formulas for some restricted partition functions $p(n|S)$ valid for all positive integers n and for some finite subset S of the natural numbers. In particular, the method of Andrews and Eriksson is modified for some conditions of elements in the set S, where $S=\{1,2,3,...,m\}$ and using the generating function, partial fraction expansion, $q-$partial fractions, the greatest integer function and the nearest integer function. The results of this research found that finding formulas for some partition functions $p(n|S)=p(n|\text{parts in } S)$ can be discovered as the set of prime numbers from 2 to 7 and the set of even numbers from 2 to 8.

Keywords: *Generating Function, Restricted Partition Functions and q-Partial Fractions.*

1. Introduction

A ***partition*** of the positive integers n is a way of writing n as a sum of positive integers where the order of the integers in the sum does not matter. We specify a partition n when we write it as a sum of positive integers such that $n_1+n_2+...+n_r=n.$ The integers $n_1,n_2,...,n_r$ are called the *parts* of partition $n.$ The number of different partitions of n is denoted by $p(n).$ We call $p(n)$ the *partition functions*. $p(0)=1$ is defined which makes sense because there is exactly one partition of the integer $0,$ the empty partition that has no parts. For example, since 10 can be expressed as the sum of positive integers by:

$10,\ 9+1,\ 8+2,\ 8+1+1,\ 7+3,\ 7+2+1,\ 7+1+1+1,\ 6+4,\ 6+3+1,$ $6+2+2,\ 6+2+1+1,\ 6+1+1+1+1,\ 5+5,\ 5+4+1,\ 5+3+2,\ 5+3+1+1,\ 5+2+2+1,\ 5+2+1+1+1,$ $5+1+1+1+1+1,\ 4+4+2,\ 4+4+1+1,\ 4+3+3,\ 4+3+2+1,\ 4+3+1+1+1,\ 4+2+2+2,\ 4+2+2+1+1,$ $4+2+1+1+1+1,\ 4+1+1+1+1+1+1,\ 3+3+3+1,\ 3+3+2+2,\ 3+3+2+1+1,\ 3+3+1+1+1+1,$ $3+2+2+2+1,\ 3+2+2+1+1+1,\ 3+2+1+1+1+1+1,\ 3+1+1+1+1+1+1+1,\ 2+2+2+2+2,$ $2+2+2+2+1+1,\ 2+2+2+1+1+1+1,\ 2+2+1+1+1+1+1+1,\ 2+1+1+1+1+1+1+1+1$ and $1+1+1+1+1+1+1+1+1+1.$ As a result,
$p(10)=42.$

A ***restricted partition*** is a partition in which some kind of restriction is imposed upon the parts. In this directions, restricted partitions are discussed in H. L. Alder (1969 & 1979) and G. E. Andrew and E. Kimmo (2004).

In 2004, Andrews and Eriksson discovered formulas for some restricted partition functions $p(n,m)$, the number of partitions of n into parts less than or equal to m, for $m = 1,2,3,4$ by using the generating functio

$$\sum_{n=0}^{\infty} p(n,m)\, q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}, \qquad (1.1)$$

Where $p(n,m) = p(n \mid \text{parts in } \{1,2,3,...,m\})$.

Therefore, the purpose of finding formulas for some $p(n \mid \text{condition})$ is interestingly studied by extended the results of Andrews and Eriksson and modified some conditions in the set of positive integers in this research.

2. Materials and Methods

In this section, we give fundamental definitions concerning the generating function, q-fraction, the greatest integer function and the nearest integer function are given to find formulas for some $p(n \mid S)$, where is a finite subset of $\mathbb{N}$

Definition 2.1 Let S be a finite subset of $\mathbb{N}$ and $p(n \mid S)$ denote the number of partitions of n into elements of S (*or in other words, the parts of partitions belong to* S). Then the univariate generating function is given by:

$$\sum_{n=0}^{\infty} p(n \mid S)\, q^n = \prod_{j \in S} \frac{1}{1-q^j}, \qquad (2.1)$$

Definition 2.2 (Munagi, 2007). Let q-fraction $v(q)/(1-q^n)^s$ is called basic if it satisfies

$$\deg(v) < \phi(n), \qquad (2.2)$$

where ϕ is the Euler $\text{phi}-\text{function}$.

Definition 2.3 (Rosen, 2015). The greatest integer in a real number x, denoted by $\lfloor x \rfloor$ is the largest integer less than or equal to x. That is $\lfloor x \rfloor$ is the integer satisfying

$$\lfloor x \rfloor \le x < \lfloor x \rfloor + 1. \qquad (2.3)$$

Definition 2.4 (Andrews and Eriksson, 2004). Let x be a positive real number and $\{x\}$ is defined to be the nearest integer to x with the convention that for any $n \in \mathbb{N}$

$$\left\{ n + \frac{1}{2} \right\} = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd}. \end{cases} \qquad (2.4)$$

3. Results

In this section, some formulas for $p(n \mid S)$ are found, where S is a finite subset of $\mathbb{N}$

Theorem 3.1 Let $n \in \mathbb{N}$ Then we have

$$p(n \mid \{2,3\}) = \left\{ \frac{2n+2}{3} - \left\lfloor \frac{n+1}{2} \right\rfloor \right\}.$$

Proof. By Eq. (2.1), we have

$$\sum_{n=0}^{\infty} p(n \mid \{2,3\})\, q^n = \frac{1}{(1-q^2)(1-q^3)} = \frac{1}{(1-q)^2(1+q)(1+q+q^2)}$$
$$= \frac{A}{(1-q)^2} + \frac{B}{1-q} + \frac{C}{1+q} + \frac{Dq+E}{(1+q+q^2)}.$$

After solving the system of linear equations by expanding the ordinary partial fraction decomposition, (3.1) is obtained as below:

$$\sum_{n=0}^{\infty} p(n \mid \{2,3\})\, q^n = \frac{1/6}{(1-q)^2} + \frac{1/4}{1-q} + \frac{1/4}{1+q} + \frac{1/3}{1+q+q^2}. \qquad (3.1)$$

Next, each summand is transformed on the right-hand side into a sum of q-fractions (the first two fractions are already basic):

$$\frac{1/4}{1+q} = \frac{(1-q)(1/4)}{(1-q)(1+q)} = \frac{-1/4}{1-q} + \frac{1/2}{1-q^2} \text{ and} \qquad (3.2)$$

$$\frac{1/3}{1+q+q^2} = \frac{(1-q)(1/3)}{(1-q)(1+q+q^2)} = \frac{(1-q)/2}{1-q^3}. \qquad (3.3)$$

Substituting the equations (3.2) and (3.3) into (3.1), we have

$$\sum_{n=0}^{\infty} p(n\mid\{2,3\})\,q^n = \frac{1/6}{(1-q)^2} + \frac{1/2}{1-q^2} + \frac{(1-q)/3}{1-q^3}.$$

By the binomial series formula of Andrews and Eriksson (2004), we have

$$\sum_{n=0}^{\infty} \binom{n+m}{m} q^n = (1-q)^{-m-1},\ |q|<1 \tag{3.4}$$

where $\binom{n+m}{m} = \frac{(n+1)(n+2)\cdots\qquad}{m!}\ \cdot = \frac{(n+m)!}{n!m!}.$

If $m=0,$ then $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q},\ |q|<1.$ (3.5)

Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} p(n\mid\{2,3\})\,q^n &= \frac{1/6}{(1-q)^2} + \frac{1/2}{1-q^2} + \frac{(1-q)/3}{1-q^3}\\ &= \frac{1}{6}\sum_{n=0}^{\infty}\binom{n+1}{1}q^n + \frac{1}{2}\sum_{n=0}^{\infty} q^{2n} + \frac{1-q}{3}\sum_{n=0}^{\infty} q^{3n}.\end{aligned}$$

By the formula of Andrews and Eriksson (2004), we have

$$\sum_{n=0}^{\infty} a_n q^{2n} = \sum_{n=0}^{\infty} a_{n/2}\left((n+1) - 2\left\lfloor\frac{n+1}{2}\right\rfloor\right) q^n \tag{3.6}$$

where $(n+1) - 2\left\lfloor\frac{n+1}{2}\right\rfloor = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 1 \text{ if } n \text{ is even}. \end{cases}$

Then,

$$\begin{aligned}\sum_{n=0}^{\infty} p(n\mid\{2,3\})\,q^n &= \frac{1}{6}\sum_{n=0}^{\infty}\binom{n+1}{1}q^n + \frac{1}{2}\sum_{n=0}^{\infty}\left((n+1) - 2\left\lfloor\frac{n+1}{2}\right\rfloor\right)q^n + \sum_{n=0}^{\infty}\left(\frac{1}{3}q^{3n} - \frac{1}{3}q^{3n+1}\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{6}\cdot\frac{(n+1)!}{n!}\right)q^n + \sum_{n=0}^{\infty}\left(\frac{(n+1)}{2} - \left\lfloor\frac{n+1}{2}\right\rfloor\right)q^n + \sum_{n=0}^{\infty}\left(\frac{1}{3}q^{3n} - \frac{1}{3}q^{3n+1}\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{(n+1)}{6} + \frac{(n+1)}{2} - \left\lfloor\frac{n+1}{2}\right\rfloor\right)q^n + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)q^{3n} + \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)q^{3n+1}\\ &= \sum_{n=0}^{\infty}\left(\frac{2n+2}{3} - \left\lfloor\frac{n+1}{2}\right\rfloor\right)q^n + \sum_{3\mid n}\left(\frac{1}{3}\right)q^n + \sum_{3\mid n-1}\left(\frac{-1}{3}\right)q^n\\ &= \sum_{n=0}^{\infty}\left(\frac{2n+2}{3} - \left\lfloor\frac{n+1}{2}\right\rfloor + \varepsilon(n)\right)q^n,\end{aligned}$$

where $\varepsilon(n) = \begin{cases} 1/3 & \text{if } n \equiv 0 \pmod 3 \\ -1/3 & \text{if } n \equiv 1 \pmod 3 \\ 0 & \text{if } n \equiv 2 \pmod 3). \end{cases}$

$\varepsilon(n)$ can be seen that it takes only the values $-\frac{1}{3}, 0, \frac{1}{3}.$ Thus, the below result is concluded by the uniqueness of Maclaurin series expansion that

$$p(n\mid\{2,3\}) = \frac{2n+2}{3} - \left\lfloor\frac{n+1}{2}\right\rfloor + \varepsilon(n).$$

Note that $p(n\mid\{2,3\})$ is an integer and $|\varepsilon(n)| < \frac{1}{2}.$

Therefore,

$$p(n\,|\,\{2,3\}) = \left\{\frac{2n+2}{3} - \left\lfloor \frac{n+1}{2} \right\rfloor\right\}.$$

This completes the proof.

Theorem 3.2 Let $n \in \mathbb{N}$ and then

$$p(n\,|\,\{2,3,5\}) \;=\; \left\{\frac{(n+1)(n+9)}{60} + \chi(n)\right\},$$

where $\chi(n) = \begin{cases} 1 & \text{if } n \equiv 0 (\bmod 5) \text{ and } (2\,|\,n \text{ or } 3 \nmid (n-1)), \\ 0 & \text{otherwise}. \end{cases}$

Proof. By Eq. (2.1), then

$$\begin{aligned}\sum_{n=0}^{\infty} p(n\,|\,\{2,3,5\})\,q^n &= \frac{1}{(1-q^2)(1-q^3)(1-q^5)} \\ &= \frac{1}{(1-q)^3(1+q)(1+q+q^2)(1+q+q^2+q^3+q^4)} \\ &= \frac{A}{(1-q)^3} + \frac{B}{(1-q)^2} + \frac{C}{1-q} + \frac{D}{1+q} + \frac{Eq+F}{1+q+q^2} + \frac{Gq^3+Hq^2+Iq+J}{1+q+q^2+q^3+q^4}.\end{aligned}$$

After solving the system of linear equations by expanding the ordinary partial fraction decomposition, then obtaining

$$\begin{aligned}\sum_{n=0}^{\infty} p(n\,|\,\{2,3,5\})\,q^n &= \frac{1/30}{(1-q)^3} + \frac{7/60}{(1-q)^2} + \frac{77/360}{1-q} + \frac{1/8}{1+q} \\ &\quad + \frac{(1-q)/9}{1+q+q^2} + \frac{(q^3+q^2+q+2)/5}{1+q+q^2+q^3+q^4}. \end{aligned} \tag{3.7}$$

Next, each summand is transformed on the right-hand side into a sum of q-fractions (the first three fractions are already basic):

$$\frac{1/8}{1+q} = \frac{(1-q)(1/8)}{(1-q)(1+q)} = \frac{-1/8}{1-q} + \frac{1/4}{1-q^2} \tag{3.8}$$

$$\frac{(1-q)/9}{1+q+q^2} = \frac{(1-q)(1-q)/9}{(1-q)(1+q+q^2)} = \frac{1/9}{1-q} - \frac{q/3}{1-q^3} \quad \text{and} \tag{3.9}$$

$$\frac{(q^3+q^2+q+2)/5}{1+q+q^2+q^3+q^4} = \frac{(1-q)(q^3+q^2+q+2)/5}{(1-q)(1+q+q^2+q^3+q^4)} = \frac{-1/5}{1-q} + \frac{(3+q^2+q^3)/5}{1-q^5}. \tag{3.10}$$

Substituting the equations (3.8), (3.9) and (3.10) into (3.7) and then

$$\sum_{n=0}^{\infty} p(n\,|\,\{2,3,5\})\,q^n = \frac{1/30}{(1-q)^3} + \frac{7/60}{(1-q)^2} - \frac{q/3}{1-q^3} + \frac{(3+q^2+q^3)/5}{1-q^5}.$$

By Eq. (3.4) and Eq. (3.5), then the below result is obtained

$$\begin{aligned}\sum_{n=0}^{\infty} p(n\,|\,\{2,3,5\})\,q^n &= \frac{1}{30}\sum_{n=0}^{\infty}\binom{n+2}{2}q^n + \frac{7}{60}\sum_{n=0}^{\infty}\binom{n+1}{1}q^n + \frac{1}{4}\sum_{n=0}^{\infty}q^{2n} \\ &\quad - \frac{q}{3}\sum_{n=0}^{\infty}q^{3n} + \frac{3+q^2+q^3}{5}\sum_{n=0}^{\infty}q^{5n} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{30}\cdot\frac{(n+2)!}{n!2!}\right)q^n + \sum_{n=0}^{\infty}\left(\frac{7}{60}\cdot\frac{(n+1)!}{n!1!}\right)q^n + \frac{1}{4}\sum_{n=0}^{\infty}q^{2n}\end{aligned}$$

$$-\frac{1}{3}\sum_{n=0}^{\infty}q^{3n+1}+\frac{3+q^2+q^3}{5}\sum_{n=0}^{\infty}q^{5n}$$

$$=\sum_{n=0}^{\infty}\left(\frac{(n+1)(n+9)}{60}\right)q^n+\sum_{2|n}\left(\frac{1}{4}\right)q^n+\sum_{3|n-1}\left(\frac{-1}{3}\right)q^n$$

$$+\sum_{5|n}\left(\frac{3}{5}\right)q^n+\sum_{5|n-2}\left(\frac{1}{5}\right)q^n+\sum_{5|n-3}\left(\frac{1}{5}\right)q^n$$

$$=\sum_{n=0}^{\infty}\left(\frac{(n+1)(n+9)}{60}+\varepsilon(n)\right)q^n, \tag{3.11}$$

Where $\varepsilon(n)=f_1(n)+f_2(n)+f_3(n)$ and

$$f_1(n)=\begin{cases}1/4 & \text{if } n\equiv 0(\text{mod}\,2)\\ 0 & \text{if } n\equiv 1(\text{mod}\,2)\end{cases}$$

$$f_2(n)=\begin{cases}-1/3 & \text{if } n\equiv 1(\text{mod}\,3)\\ 0 & \text{if } n\equiv 0,2(\text{mod}\,3)\end{cases}$$

$$f_3(n)=\begin{cases}3/5 & \text{if } n\equiv 0(\text{mod}\,5)\\ 1/5 & \text{if } n\equiv 2,3(\text{mod}\,5)\\ 0 & \text{if } n\equiv 1,4(\text{mod}\,5).\end{cases}$$

There are 12 possible values of $\varepsilon(n)$, namely:

$$\frac{17}{20},\frac{-1}{3},\frac{9}{20},\frac{-1}{12},\frac{-2}{15},\frac{3}{5},0,\frac{7}{60},\frac{4}{15},\frac{1}{4},\frac{1}{5},\frac{31}{60},$$

With

$|\varepsilon(n)|>\frac{1}{2}$ if and only if $n\equiv 0(\text{mod}\,5)$ and $(2\,|\,n$ or $3\nmid(n-1))$.

By comparing the coefficients of the power series in Eq. (3.11), then

$$p(n\,|\,\{2,3,5\})=\frac{(n+1)(n+9)}{60}+\varepsilon(n).$$

Define

$$\chi(n)=\begin{cases}1 & \text{if } n\equiv 0(\text{mod}\,5)\,\text{and}\,(2\,|\,n \text{ or } 3\nmid(n-1)),\\ 0 & \text{otherwise}.\end{cases}$$

Since $p(n\,|\,\{2,3,5\})$ is a non-negative integer, then the below result is obtained

$$p(n\,|\,\{2,3,5\})=\left\{\frac{(n+1)(n+9)}{60}+\varepsilon(n)\right\}.$$

This completes the proof.

Theorem 3.3 Let $n\in\mathbb{N}$ and then

$$p(n\,|\,\{2,3,5,7\}) = \left\{\frac{(n+1)(2n^2+49n+656)}{2520}-\frac{1}{4}\left\lfloor\frac{n+1}{2}\right\rfloor+\chi(n)\right\},$$

where $\chi(n)=\begin{cases}1 & \text{if } n\not\equiv 1,6(\text{mod}\,7)\,\text{and}\,(n\not\equiv 2(\text{mod}\,3) \text{ or } n\not\equiv 4(\text{mod}\,5)),\\ 0 & \text{otherwise}.\end{cases}$

Proof. By Eq. (2.1) and then

$$\sum_{n=0}^{\infty}p(n\,|\,\{2,3,5,7\})\,q^n=\frac{1}{(1-q^2)(1-q^3)(1-q^5)(1-q^7)}$$

$$
\begin{aligned}
&=\frac{1}{(1-q)^4(1+q)(1+q+q^2)(1+q+q^2+q^3+q^4)(1+q+q^2+q^3+q^4+q^5+q^6)}\\
&=\frac{A}{(1-q)^4}+\frac{B}{(1-q)^3}+\frac{C}{(1-q)^2}+\frac{D}{1-q}+\frac{E}{1+q}\\
&+\frac{Fq+G}{1+q+q^2}+\frac{Hq^3+Iq^2+Jq+K}{1+q+q^2+q^3+q^4}+\frac{Lq^5+Mq^4+Nq^3+Pq^2+Qq+R}{1+q+q^2+q^3+q^4+q^5+q^6}
\end{aligned}
$$

After solving the system of linear equations by expanding the ordinary partial fraction decomposition, the below result is obtained

$$
\begin{aligned}
\sum_{n=0}^{\infty}p(n\,|\,\{2,3,5,7\})\,q^n &=\frac{1/210}{(1-q)^4}+\frac{13/420}{(1-q)^3}+\frac{251/1520}{(1-q)^2}+\frac{23/112}{1-q}+\frac{1/16}{1+q}+\frac{1/9}{1+q+q^2}\\
&+\frac{(1+q^2)/5}{1+q+q^2+q^3+q^4}+\frac{(2+q+q^2+2q^3+q^5)/7}{1+q+q^2+q^3+q^4+q^5+q^6}
\end{aligned}
\tag{3.12}
$$

Next, each summand is transformed on the right-hand side into a sum of q-fractions (the first four fractions are already basic):

$$\frac{1/16}{1+q}=\frac{(1-q)(1/16)}{(1-q)(1+q)}=\frac{-1/16}{1-q}+\frac{1/8}{1-q^2}\tag{3.13}$$

$$\frac{1/9}{1+q+q^2}=\frac{(1-q)/9}{(1-q)(1+q+q^2)}=\frac{(1-q)/9}{1-q^3}\tag{3.14}$$

$$\frac{(1+q^2)/5}{1+q+q^2+q^3+q^4}=\frac{(1-q+q^2-q^3)/5}{1-q^5}\tag{3.15}$$

$$\frac{(2+q+q^2+2q^3+q^5)/7}{1+q+q^2+q^3+q^4+q^5+q^6}=\frac{-1/7}{1-q}+\frac{(3+q^2+2q^3-q^4+2q^5)/7}{1-q^7}\tag{3.16}$$

Substituting the equations (3.13), (3.14), (3.15) and (3.16) into (3.12) then

$$
\begin{aligned}
\sum_{n=0}^{\infty}p(n\,|\,\{2,3,5,7\})\,q^n &=\frac{1/210}{(1-q)^4}+\frac{13/420}{(1-q)^3}+\frac{251/1520}{(1-q)^2}+\frac{1/8}{1-q^2}+\frac{(1-q)/9}{1-q^3}\\
&+\frac{(1-q+q^2-q^3)/5}{1-q^5}+\frac{(3+q^2+2q^3-q^4+2q^5)/7}{1-q^7}.
\end{aligned}
$$

By Eq. (3.4), Eq. (3.5) and Eq. (3.6), then the below result is obtained

$$
\begin{aligned}
\sum_{n=0}^{\infty}p(n\,|\,\{2,3,5,7\})\,q^n &=\frac{1}{210}\sum_{n=0}^{\infty}\binom{n+3}{3}q^n+\frac{13}{420}\sum_{n=0}^{\infty}\binom{n+2}{2}q^n+\frac{250}{2520}\sum_{n=0}^{\infty}\binom{n+1}{1}q^n+\frac{1}{8}\sum_{n=0}^{\infty}q^{2n}\\
&+\frac{1-q}{9}\sum_{n=0}^{\infty}q^{3n}+\frac{1-q+q^2-q^3}{5}\sum_{n=0}^{\infty}q^{5n}+\frac{3+q^2+2q^3-q^4+2q^5}{7}\sum_{n=0}^{\infty}q^{7n}\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{120}\cdot\frac{(n+3)!}{n!3!}\right)q^n+\sum_{n=0}^{\infty}\left(\frac{13}{240}\cdot\frac{(n+2)!}{n!2!}\right)q^n+\sum_{n=0}^{\infty}\left(\frac{251}{2520}\cdot\frac{(n+1)!}{n!1!}\right)q^n\\
&+\frac{1}{8}\sum_{n=0}^{\infty}\left((n+1)-2\left\lfloor\frac{n+1}{2}\right\rfloor\right)q^n+\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)q^{3n}+\sum_{n=0}^{\infty}\left(\frac{-1}{9}\right)q^{3n+1}+\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)q^{5n}\\
&+\sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)q^{5n+1}+\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)q^{5n+2}+\sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)q^{5n+3}+\sum_{n=0}^{\infty}\left(\frac{3}{7}\right)q^{7n}+\sum_{n=0}^{\infty}\left(\frac{1}{7}\right)q^{7n+2}\\
&+\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)q^{7n+3}+\sum_{n=0}^{\infty}\left(\frac{-1}{7}\right)q^{7n+4}+\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)q^{7n+5}
\end{aligned}
$$

$$=\sum_{n=0}^{\infty}\left(\frac{(n+1)(2n^2+49n+341)}{2520}+\left(\frac{(n+1)}{8}-\frac{1}{4}\left\lfloor\frac{n+1}{2}\right\rfloor\right)\right)q^n$$
$$+\sum_{3|n}\left(\frac{1}{9}\right)q^n+\sum_{3|n-1}\left(\frac{-1}{9}\right)q^n+\sum_{5|n}\left(\frac{1}{5}\right)q^n+\sum_{5|n-1}\left(\frac{-1}{5}\right)q^n$$
$$+\sum_{5|n-2}\left(\frac{1}{5}\right)q^n+\sum_{5|n-3}\left(\frac{-1}{5}\right)q^n+\sum_{7|n}\left(\frac{3}{7}\right)q^n+\sum_{7|n-2}\left(\frac{1}{7}\right)q^n$$
$$+\sum_{7|n-3}\left(\frac{2}{7}\right)q^n+\sum_{7|n-4}\left(\frac{-1}{7}\right)q^n+\sum_{7|n-5}\left(\frac{2}{7}\right)q^n$$
$$=\sum_{n=0}^{\infty}\left(\frac{(n+1)(2n^2+49n+656)}{2520}-\frac{1}{4}\left\lfloor\frac{n+1}{2}\right\rfloor+\varepsilon(n)\right)q^n, \tag{3.17}$$

Where $\varepsilon(n)=f_1(n)+f_2(n)+f_3(n)$ and

$$f_1(n)=\begin{cases}1/9 & \text{if } n\equiv 0 \pmod 3\\ -1/9 & \text{if } n\equiv 1 \pmod 3\\ 0 & \text{if } n\equiv 2 \pmod 3),\end{cases}$$
$$f_2(n)=\begin{cases}1/5 & \text{if } n\equiv 0,2 \pmod 5\\ -1/5 & \text{if } n\equiv 1,3 \pmod 5\\ 0 & \text{if } n\equiv 4 \pmod 5),\end{cases}$$
$$f_3(n)=\begin{cases}3/7 & \text{if } n\equiv 0 \pmod 7\\ 1/7 & \text{if } n\equiv 2 \pmod 7\\ 2/7 & \text{if } n\equiv 3,5 \pmod 7\\ -1/7 & \text{if } n\equiv 4 \pmod 7\\ 0 & \text{if } n\equiv 1,6 \pmod 7).\end{cases}$$

There are 46 possible values of $\varepsilon(n)$, namely:

$$\frac{233}{315},\frac{-14}{45},\frac{12}{35},\frac{62}{315},\frac{-61}{63},\frac{17}{35},\frac{-4}{45},\frac{163}{315},\frac{-1}{5},\frac{16}{63},\frac{118}{315},\frac{-12}{35},\frac{188}{315},\frac{3}{7},$$
$$\frac{14}{45},\frac{-53}{315},\frac{-73}{315},\frac{11}{63},\frac{1}{5},\frac{107}{315},\frac{4}{45},\frac{-2}{35},\frac{25}{63},\frac{-17}{315},\frac{3}{35},\frac{37}{315},0,\frac{143}{315},\frac{-8}{35},\frac{2}{35},$$
$$\frac{-1}{9},\frac{22}{35},\frac{73}{315},\frac{-2}{63},\frac{1}{7},\frac{-143}{315},\frac{20}{63},\frac{17}{315},\frac{8}{35},\frac{2}{7},\frac{53}{315},\frac{1}{9},\frac{163}{315},\frac{-1}{7},\frac{2}{63},\frac{34}{63}$$

with

$$|\varepsilon(n)|>\frac{1}{2} \text{ if and only if } n\not\equiv 1,6 \pmod 7 \text{ and } (n\not\equiv 2 \pmod 3) \text{ or } n\not\equiv 4 \pmod 5)).$$

By comparing the coefficients of the power series in Eq. (3.17), the below result is obtained

$$p(n\,|\,\{2,3,5,7\})=\frac{(n+1)(2n^2+49n+656)}{2520}-\frac{1}{4}\left\lfloor\frac{n+1}{2}\right\rfloor+\varepsilon(n).$$

Define

$$\chi(n)=\begin{cases}1 & \text{if } n\not\equiv 1,6 \pmod 7 \text{ and } (n\not\equiv 2 \pmod 3) \text{ or } n\not\equiv 4 \pmod 5)),\\ 0 & \text{otherwise}.\end{cases}$$

Since $p(n\,|\,\{2,3,5,7\})$ is a non-negative integer, then this below result is presented

$$p(n\,|\,\{2,3,5,7\}) = \left\{ \frac{(n+1)(2n^2+49n+656)}{2520} - \frac{1}{4}\left\lfloor \frac{n+1}{2} \right\rfloor + \varepsilon(n) \right\}.$$

This completes the proof.

Theorem 3.4 Let $n \in \mathbb{N}$ and than

$$p(n\,|\,\{2,4\}) = \left(\frac{n+2}{2}\right)\left(\frac{n+1}{2} - \left\lfloor \frac{n+1}{2} \right\rfloor\right) + \frac{1}{2}\left(\left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor\right).$$

Proof. By Eq. (2.1) and then

$$\sum_{n=0}^{\infty} p(n\,|\,\{2,4\})\,q^n = \frac{1}{(1-q^2)(1-q^4)} = \frac{1}{(1-q)^2(1+q)^2(1+q^2)}$$
$$= \frac{A}{(1-q)^2} + \frac{B}{1-q} + \frac{C}{(1+q)^2} + \frac{D}{1+q} + \frac{Eq+F}{1+q^2}.$$

After solving the system of linear equations by expanding the ordinary partial fraction decomposition, (3.18) is obtained

$$\sum_{n=0}^{\infty} p(n\,|\,\{2,4\})\,q^n = \frac{1/8}{(1-q)^2} + \frac{1/4}{1-q} + \frac{1/8}{(1+q)^2} + \frac{1/4}{1+q} + \frac{1/4}{1+q^2}. \tag{3.18}$$

Next, each summand is transformed on the right-hand side into a sum of q-fractions (the first two fractions are already basic):

$$\frac{1/4}{1+q} = \frac{(1-q)(1/4)}{(1-q)(1+q)} = \frac{-1/4}{1-q} + \frac{1/2}{1-q^2} \tag{3.19}$$

$$\frac{1/4}{1+q^2} = \frac{(1-q^2)/4}{(1-q^2)(1+q^2)} = \frac{-1/4}{1-q^2} + \frac{1/2}{1-q^4} \tag{3.20}$$

$$\frac{1/8}{(1+q)^2} = \frac{(1-q^2)/8}{(1-q)^2(1+q)^2} = \frac{-1/8}{(1-q)^2} - \frac{1/4}{1-q^4} + \frac{1/2}{(1-q^2)^2} \tag{3.21}$$

Substituting the equations (3.19), (3.20) and (3.21) into (3.18), we have

$$\sum_{n=0}^{\infty} p(n\,|\,\{2,4\})\,q^n = \frac{1/2}{(1-q^2)^2} + \frac{1/2}{1-q^4}.$$

By Eq. (3.4) and Eq. (3.5), the below result is obtained

$$\sum_{n=0}^{\infty} p(n\,|\,\{2,4\})\,q^n = \frac{1}{2}\sum_{n=0}^{\infty}\binom{n+1}{1}q^{2n} + \frac{1}{2}\sum_{n=0}^{\infty} q^{4n} = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)\,q^{2n} + \frac{1}{2}\sum_{n=0}^{\infty} q^{4n}.$$

By Eq. (3.6), then

$$\sum_{n=0}^{\infty} a_n q^{2n} = \sum_{n=0}^{\infty} a_{n/2}\left((n+1) - 2\left\lfloor \frac{n+1}{2} \right\rfloor\right) q^n,$$

Where $(n+1) - 2\left\lfloor \frac{n+1}{2} \right\rfloor = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even}. \end{cases}$

By similar reasons, from the above one then (3.12) is obtained

$$\sum_{n=0}^{\infty} a_n q^{4n} = \sum_{n=0}^{\infty} a_{n/4}\left(\left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor\right) q^n, \tag{3.22}$$

where $\left\lfloor \frac{n}{4} \right\rfloor - \left\lfloor \frac{n-1}{4} \right\rfloor = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{if } n \not\equiv 0 \pmod 4. \end{cases}$

Then,

$$\sum_{n=0}^{\infty} p(n\,|\,\{2,4\})\,q^n = \sum_{n=0}^{\infty}\left(\frac{1}{2}\left(\frac{n}{2}+1\right)\left((n+1)-2\left\lfloor\frac{n+1}{2}\right\rfloor\right)\right)q^n$$
$$+\sum_{n=0}^{\infty}\left(\frac{1}{2}\left(\left\lfloor\frac{n}{4}\right\rfloor-\left\lfloor\frac{n+1}{4}\right\rfloor\right)\right)q^n$$
$$=\sum_{n=0}^{\infty}\left(\left(\frac{n+2}{2}\right)\left(\frac{n+1}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)+\frac{1}{2}\left(\left\lfloor\frac{n}{4}\right\rfloor-\left\lfloor\frac{n-1}{4}\right\rfloor\right)\right)q^n \tag{3.23}$$

By comparing the coefficients of the power series in Eq. (3.23), the below result is obtained

$$p(n\,|\,\{2,4\}) = \left(\frac{n+2}{2}\right)\left(\frac{n+1}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)+\frac{1}{2}\left(\left\lfloor\frac{n}{4}\right\rfloor-\left\lfloor\frac{n-1}{4}\right\rfloor\right).$$

This completes the proof.

Theorem 3.5 Let $n \in \mathbb{N}$ and then

$$p(n\,|\,\{2,4,6\}) = \left\{\frac{(n+6)^2}{24}\left(\frac{n+1}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)\right\}.$$

Proof. By Eq. (2.1) and then

$$\sum_{n=0}^{\infty} p(n\,|\,\{2,4,6\})\,q^n = \frac{1}{(1-q^2)(1-q^4)(1-q^6)}$$
$$= \frac{1}{(1-q)^3(1+q)^3(1+q^2)(1+q+q^2)(1-q+q^2)}$$
$$= \frac{A}{(1-q)^3}+\frac{B}{(1-q)^2}+\frac{C}{1-q}+\frac{D}{(1+q)^3}+\frac{E}{(1+q)^2}+\frac{F}{1+q}$$
$$+\frac{Gq+H}{1+q^2}+\frac{Iq+J}{1+q+q^2}+\frac{Kq+L}{1-q+q^2}.$$

After solving the system of linear equations by expanding the ordinary partial fraction decomposition, (3.24) is obtained

$$\sum_{n=0}^{\infty} p(n\,|\,\{2,4,6\})\,q^n = \frac{1/48}{(1-q)^3}+\frac{3/32}{(1-q)^2}+\frac{61/288}{1-q}+\frac{1/4}{(1+q)^3}+\frac{3/32}{(1+q)^2}+\frac{61/288}{1+q}$$
$$+\frac{1/8}{1+q^2}+\frac{(2+q)/18}{1+q+q^2}+\frac{(2-q)/18}{1-q+q^2}. \tag{3.24}$$

Next, each summand is transformed on the right-hand side into a sum of q-fractions (the first three fractions are already basic):

$$\frac{1/8}{1+q^2} = \frac{(1-q^2)(1/4)}{(1-q^2)(1+q^2)} = \frac{-1/8}{1-q^2}+\frac{1/4}{1-q^4} \tag{3.25}$$

$$\frac{61/288}{1+q} = \frac{(1-q)(61/288)}{(1-q)(1+q)} = \frac{-61/288}{1-q}+\frac{61/144}{1-q^2} \tag{3.26}$$

$$\frac{(2+q)/18}{1+q+q^2} = \frac{(1-q)(2+q)(1/18)}{(1-q)(1+q+q^2)} = \frac{-1/18}{1-q}+\frac{1/6}{1-q^3} \tag{3.27}$$

$$\frac{3/32}{(1+q)^2} = \frac{(1-q)^2(3/32)}{(1-q)^2(1+q)^2} = \frac{-3/32}{(1-q)^2}+\frac{3/8}{(1-q^2)^2}-\frac{3/16}{1-q^2} \tag{3.28}$$

$$\frac{1/48}{(1+q)^3} = \frac{(1-q)^3(1/48)}{(1-q)^3(1+q)^3} = \frac{-1/48}{(1-q)^3}+\frac{1/6}{(1-q^2)^3}-\frac{1/8}{(1-q^2)^2} \tag{3.29}$$

$$\frac{(2-q)/18}{1-q+q^2}=\frac{(2-q)(1+q)(1/18)}{(1+q)(1-q+q^2)}=\frac{1/18}{1-q}-\frac{1/9}{1-q^2}-\frac{1/6}{1-q^3}+\frac{1/3}{1-q^6} \tag{3.30}$$

Substituting the equations (3.25), (3.26), (3.27), (3.28), (3.29) and (3.30) into (3.24), then

$\sum_{n=0}^{\infty}p(n\,|\,\{2,4,6\})\,q^n=\frac{1/6}{(1-q^2)^3}+\frac{1/4}{(1-q^2)^2}+\frac{1/4}{1-q^4}+\frac{1/3}{1-q^6}$ is obtained.

By Eq. (3.4) and Eq. (3.5), then

$$\begin{aligned}\sum_{n=0}^{\infty}p(n\,|\,\{2,4,6\})\,q^n&=\frac{1}{6}\sum_{n=0}^{\infty}\binom{n+2}{2}q^{2n}+\frac{1}{4}\sum_{n=0}^{\infty}\binom{n+1}{1}q^{2n}+\frac{1}{4}\sum_{n=0}^{\infty}q^{4n}+\frac{1}{3}\sum_{n=0}^{\infty}q^{6n}\\&=\sum_{n=0}^{\infty}\left(\frac{(n+1)(n+5)}{12}\right)q^{2n}+\frac{1}{4}\sum_{n=0}^{\infty}q^{4n}+\frac{1}{3}\sum_{n=0}^{\infty}q^{6n}\\&=\sum_{n=0}^{\infty}\left(\frac{n^2+6n+9}{12}-\frac{4}{12}\right)q^{2n}+\frac{1}{4}\sum_{n=0}^{\infty}q^{4n}+\frac{1}{3}\sum_{n=0}^{\infty}q^{6n}\\&=\sum_{n=0}^{\infty}\left(\frac{(n+3)^2}{12}\right)q^{2n}+\frac{1}{4}\sum_{n=0}^{\infty}q^{4n}+\sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)q^{2n}+\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)q^{6n}.\end{aligned}$$

By Eq. (3.6). Then,

$$\begin{aligned}\sum_{n=0}^{\infty}p(n\,|\,\{2,4,6\})\,q^n&=\sum_{n=0}^{\infty}\left(\frac{(n+6)^2}{48}\left((n+1)-2\left\lfloor\frac{n+1}{2}\right\rfloor\right)\right)q^n+\sum_{2|n}\left(\frac{-1}{3}\right)q^n+\sum_{4|n}\left(\frac{1}{4}\right)q^n\sum_{6|n}\left(\frac{1}{3}\right)q^n\\&=\sum_{n=0}^{\infty}\left(\frac{(n+6)^2}{48}\left(\frac{(n+1)}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)+\varepsilon(n)\right)q^n,\end{aligned} \tag{3.31}$$

where $\varepsilon(n)=f_1(n)+f_2(n)+f_3(n)$ and

$$f_1(n)=\begin{cases}-1/3 & \text{if } n\equiv 0(\operatorname{mod}2)\\ 0 & \text{if } n\equiv 1(\operatorname{mod}2),\end{cases}$$

$$f_2(n)=\begin{cases}1/4 & \text{if } n\equiv 0(\operatorname{mod}4)\\ 0 & \text{if } n\equiv 1,2,3(\operatorname{mod}4),\end{cases}$$

$$f_3(n)=\begin{cases}1/3 & \text{if } n\equiv 0(\operatorname{mod}6)\\ 0 & \text{if } n\equiv 1,2,3,4,5(\operatorname{mod}6).\end{cases}$$

We see that $\varepsilon(n)$ takes only the values

$$\frac{-1}{12},\ \frac{-1}{3},\ 0,\ \frac{1}{4}.$$

Now the below result can be concluded by the uniqueness of Maclaurin series expansions as

$$p(n\,|\,\{2,4,6\})=\frac{(n+6)^2}{48}\left(\frac{(n+1)}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)+\varepsilon(n).$$

By $p(n\,|\,\{2,4,6\})$ is a positive integer and $|\varepsilon(n)|<\frac{1}{2}$.

Therefore,

$$p(n\,|\,\{2,4,6\})=\left\{\frac{(n+6)^2}{48}\left(\frac{(n+1)}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)\right\}.$$

This completes the proof.

Theorem 3.6 Let $n\in\mathbb{N}$ and then

$$p(n\,|\,\{2,4,6,8\})=\left\{\left(\frac{(n+2)(n+14)^2+128}{576}\right)\left(\frac{n+1}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)\right.$$
$$\left.+\left(\frac{n+6}{32}\right)\left(\left\lfloor\frac{n}{4}\right\rfloor-\left\lfloor\frac{n-1}{4}\right\rfloor\right)\right\}.$$

Proof. By Eq. (2.1), then

$$\sum_{n=0}^{\infty}p(n\,|\,\{2,4,6,8\})\,q^n=\frac{1}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)}$$
$$=\frac{1}{(1-q)^4(1+q)^4(1+q^2)^2(1+q^4)(1+q+q^2)(1-q+q^2)}$$
$$=\frac{A}{(1-q)^4}+\frac{B}{(1-q)^3}+\frac{C}{(1-q)^2}+\frac{D}{1-q}+\frac{E}{(1+q)^4}+\frac{F}{(1+q)^3}+\frac{J}{(1+q)^2}+\frac{H}{1+q}$$
$$+\frac{Iq+J}{(1+q^2)^2}+\frac{Kq+L}{1+q^2}+\frac{Mq+N}{1+q+q^2}+\frac{Oq+P}{1-q+q^2}+\frac{Qq^3+Rq^2+Sq+T}{1+q^4}.$$

After solving the system of linear equations by expanding the ordinary partial fraction decomposition, the below result is obtained

$$\sum_{n=0}^{\infty}p(n\,|\,\{2,4,6,8\})\,q^n=\frac{1/384}{(1-q)^4}+\frac{1/48}{(1-q)^3}+\frac{187/2304}{(1-q)^2}+\frac{51/256}{1-q}+\frac{1/384}{(1+q)^4}+\frac{1/48}{(1+q)^3}$$
$$+\frac{187/2304}{(1+q)^2}+\frac{51/256}{1+q}+\frac{1/32}{(1+q^2)^2}+\frac{1/8}{1+q^2}+\frac{1/18}{1+q+q^2}+\frac{1/18}{1-q+q^2}+\frac{1/8}{1+q^4} \tag{3.32}$$

Next, each summand is transformed on the right-hand side into a sum of q-fractions (the first four fractions are already basic):

$$\frac{1/8}{1+q^2}=\frac{(1-q^2)(1/8)}{(1-q^2)(1+q^2)}=\frac{-1/8}{1-q^2}+\frac{1/4}{1-q^4} \tag{3.33}$$

$$\frac{1/8}{1+q^4}=\frac{(1-q^4)(1/8)}{(1-q^4)(1+q^4)}=\frac{-1/8}{1-q^4}+\frac{1/4}{1-q^8} \tag{3.34}$$

$$\frac{1/18}{1+q+q^2}=\frac{(1-q)(1/18)}{(1-q)(1+q+q^2)}=\frac{(1-q)/18}{1-q^3} \tag{3.35}$$

$$\frac{51/256}{1+q}=\frac{(1-q)(51/256)}{(1-q)(1+q)}=\frac{-51/256}{1-q}+\frac{51/128}{1-q^2} \tag{3.36}$$

$$\frac{1/48}{(1+q)^3}=\frac{(1-q)^3(1/48)}{(1-q)^3(1+q)^3}=\frac{-1/48}{(1-q)^3}-\frac{1/8}{(1-q^2)^2}+\frac{1/6}{(1-q^2)^3} \tag{3.37}$$

$$\frac{1/32}{(1+q^2)^2}=\frac{(1-q^2)^2(1/32)}{(1-q^2)^2(1+q^2)^2}=\frac{-1/32}{(1-q^2)^2}+\frac{1/8}{(1-q^4)^2}-\frac{1/16}{1-q^4} \tag{3.38}$$

$$\frac{1/384}{(1+q)^4}=\frac{(1-q)^4(1/384)}{(1-q)^4(1+q)^4}=\frac{-1/384}{(1-q)^4}+\frac{1/24}{(1-q^2)^4}-\frac{1/24}{(1-q^2)^3}+\frac{1/192}{(1-q^2)^2} \tag{3.39}$$

$$\frac{187/2304}{(1+q)^2}=\frac{(1-q)^2(187/2304)}{(1-q)^2(1+q)^2}=\frac{-187/2304}{(1-q)^2}-\frac{187/1152}{1-q^2}+\frac{187/576}{(1-q^2)^2} \tag{3.40}$$

$$\frac{1/18}{1-q+q^2}=\frac{(1+q)(1/18)}{(1+q)(1-q+q^2)}=\frac{(1-q^3)(1+q)(1/18)}{(1-q^3)(1+q^3)}=\frac{(-1-q)/18}{1-q^3}+\frac{(1+q)/9}{1-q^6} \tag{3.41}$$

Substituting the equations (3.33), (3.34), (3.35), (3.36), (3.367), (3.38), (3.39), (3.40) and (3.41) into (3.32), then

$$\sum_{n=0}^{\infty} p(n\,|\,\{2,4,6,8\})\,q^n = \frac{1/24}{(1-q^2)^4}+\frac{1/8}{(1-q^2)^3}+\frac{25/144}{(1-q^2)^2}+\frac{1/8}{(1-q^4)^2}+\frac{1/9}{1-q^2}-\frac{q/9}{1-q^3}$$
$$+\frac{1/16}{1-q^4}+\frac{(1+q)/9}{1-q^6}+\frac{1/4}{1-q^8}\text{ id obtained.}$$

By Eq. (3.4) and Eq. (3.5) and following that

$$\sum_{n=0}^{\infty} p(n\,|\,\{2,4,6,8\})\,q^n = \frac{1}{24}\sum_{n=0}^{\infty}\binom{n+3}{3}q^{2n}+\frac{1}{4}\sum_{n=0}^{\infty}\binom{n+2}{2}q^{2n}+\frac{25}{144}\sum_{n=0}^{\infty}\binom{n+1}{1}q^{2n}$$
$$+\frac{1}{8}\sum_{n=0}^{\infty}\binom{n+1}{1}q^{4n}+\frac{1}{9}\sum_{n=0}^{\infty}q^{2n}-\frac{1}{9}\sum_{n=0}^{\infty}q^{3n+1}+\frac{1}{16}\sum_{n=0}^{\infty}q^{4n}+\frac{1}{9}\sum_{n=0}^{\infty}q^{6n}$$
$$+\frac{1}{9}\sum_{n=0}^{\infty}q^{6n+1}+\frac{1}{4}\sum_{n=0}^{\infty}q^{8n}$$
$$=\sum_{n=0}^{\infty}\left(\frac{(n+1)(n+7)^2}{144}+\frac{1}{9}\right)q^{2n}+\sum_{n=0}^{\infty}\left(\frac{n+1}{8}+\frac{1}{16}\right)q^{4n}$$
$$+\sum_{3|n-1}\left(\frac{-1}{9}\right)q^n+\sum_{8|n}\left(\frac{1}{4}\right)q^n+\sum_{6|n}\left(\frac{1}{9}\right)q^n++\sum_{6|n-1}\left(\frac{1}{9}\right)q^n.$$

By Eq. (3.6) and Eq. (3.22), then

$$\sum_{n=0}^{\infty} p(n\,|\,\{2,4,6,8\})\,q^n = \sum_{n=0}^{\infty}\left(\left(\frac{(n+2)(n+14)^2+128}{576}\right)\left(\frac{n+1}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)\right)q^n$$
$$+\sum_{n=0}^{\infty}\left(\left(\frac{n+6}{32}\right)\left(\left\lfloor\frac{n}{4}\right\rfloor-\left\lfloor\frac{n-1}{4}\right\rfloor\right)+\varepsilon(n)\right)q^n,$$

where $\varepsilon(n)=f_1(n)+f_2(n)+f_3(n)$ and

$$f_1(n)=\begin{cases}-1/9 & \text{if } n\equiv 1 \pmod 3\\ 0 & \text{if } n\equiv 0,2 \pmod 3),\end{cases}$$

$$f_2(n)=\begin{cases}1/9 & \text{if } n\equiv 0,1 \pmod 6\\ 0 & \text{if } n\equiv 2,3,4,5 \pmod 6),\end{cases}$$

$$f_3(n)=\begin{cases}1/4 & \text{if } n\equiv 0 \pmod 8\\ 0 & \text{if } n\equiv 1,2,3,4,5,6,7 \pmod 8).\end{cases}$$

We see that $\varepsilon(n)$ can be seen that it takes only the values

$$\frac{-1}{9},\ 0,\ \frac{1}{9},\ \frac{1}{4},\ \frac{5}{36},\ \frac{13}{36}.$$

Now, the below result is concluded by the uniqueness of Maclaurin series expansions as

$$p(n\,|\,\{2,4,6,8\})=\left(\frac{(n+2)(n+14)^2+128}{576}\right)\left(\frac{n+1}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)$$
$$+\left(\frac{n+6}{32}\right)\left(\left\lfloor\frac{n}{4}\right\rfloor-\left\lfloor\frac{n-1}{4}\right\rfloor\right)+\varepsilon(n).$$

Since $p(n\,|\,\{2,4,6,8\})$ is a positive integer and $|\varepsilon(n)|<\frac{1}{2}.$ Therefore,

$$p(n\,|\,\{2,4,6,8\})=\left\{\left(\frac{(n+2)(n+14)^2+128}{576}\right)\left(\frac{n+1}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)\right.$$
$$\left.+\left(\frac{n+6}{32}\right)\left(\left\lfloor\frac{n}{4}\right\rfloor-\left\lfloor\frac{n-1}{4}\right\rfloor\right)\right\}.$$

This completes the proof.

4. Discussion

4.1 The results of this research are close to the research of G. E. Andrews and E. Kimmo (2004), who studied **formulas for partition functions** and obtained the formulas for $p(n,m)$, the number of partitions of n into parts less than or equal to m, for $m=1,2,3,4.$ In this research, similar result found in this research is to use the generating function, partial fractions decomposition, binomial series, summation of the geometric series and uniqueness of the Maclaurin series expansions, however, the main results of this research are difference because some relevant theories such as q-fractions, the greatest integer function, the nearest integer function and congruent are used to find formulas for some partition functions $p(n\,|\,S)$, where S is a finite subset of $\mathbb{N}$ that is, the set of prime numbers from 2 to 7 and the set of even numbers from 2 to 8.

4.2 The results of this research are close to the research of M. S. Ladan, D. Singh, and Y. Tella (2018), who studied **the extension of formulas for partition functions** of G. E. Andrews and E. Kimmo (2004), for $m=5,6,7,8,9,10,11$. Similar result found in this research is to use the generating function, partial fractions decomposition to obtain power series expansions, however, the main results of this research are difference because some relevant theories such as q-fractions, the greatest integer function, the nearest integer function, binomial series, summation of the geometric series, congruent and uniqueness of the Maclaurin series expansions are used to find formulas for some partition functions $p(n\,|\,S)$, where S is the set of prime numbers from 2 to 7 and the set of even numbers from 2 to 8.

4.3 The results of this research are close to the research of Augustine O. Munagi (2007), who studied the **Computation of *q*-Partial Fractions** and extended the results of the research of P.A. MacMahon (1960) and A. Cayley (1898) for $p(n,m)$, where $m=2,3$ and discovered the formulas for $p(n,m)$, where $m=4,5$. Similar result found in this research is to use the generating function, partial fractions decomposition and some definitions of the q-Partial $v(q)/(1-q^n)^s$ is called basic if it satisfies $\text{degree}(v)<\phi(n)$, where ϕ is Euler phi-function and the q-partial fraction decomposition of the q-fraction $A(q)$ is a representation of $A(q)$ as a finite sum of basic q-fractions with distinct denominators, however, the main results of this research are difference because some relevant theories such as the greatest integer function, the nearest integer function, binomial series, summation of the geometric series, congruent and uniqueness of the Maclaurin series expansions to find formulas for $p(n\,|\,S)$, where S is the set of prime numbers and set of even numbers, that is, $S=\{2,3\}$, $S=\{2,3,5\}$, $S=\{2,3,5,7\}$ and $S=\{2,4\}$, $S=\{2,4,6\}$, $S=\{2,4,6,8\}$ respectively.

5. Conclusion

The main objective of this study of some new partition function concerning restricted partition valid for all positive integers n of the general type: $p(n\,|\,S)=p(n\,|\,\text{parts in}\,S)$ can be studied by using the method of Andrews and Eriksson (2004), the generating functions and q-fractions are the tools to find formulas for some restricted partition functions. The following results is obtained

5.1 Let $n\in\mathbb{N}$ and then

$$p(n\,|\,\{2,3\})=\left\{\frac{2n+2}{3}-\left\lfloor\frac{n+1}{2}\right\rfloor\right\}.$$

5.2 Let $n\in\mathbb{N}$ and then

$$p(n\,|\,\{2,3,5\}) \;=\; \left\{\frac{(n+1)(n+9)}{60}+\chi(n)\right\},$$

where $\chi(n)=\begin{cases}1 & \text{if } n\equiv 0(\bmod 5)\,\text{and}\,(2\,|\,n \text{ or } 3\nmid(n-1)),\\ 0 & \text{otherwise}.\end{cases}$

5.3 Let $n\in\mathbb{N}$ and then

$$p(n\,|\,\{2,3,5,7\}) \;=\; \left\{\frac{(n+1)(2n^2+49n+656)}{2520}-\frac{1}{4}\left\lfloor\frac{n+1}{2}\right\rfloor+\chi(n)\right\},$$

where $\chi(n)=\begin{cases}1 & \text{if } n\not\equiv 1,6(\bmod 7)\,\text{and}\,(n\not\equiv 2(\bmod 3) \text{ or } n\not\equiv 4(\bmod 5)),\\ 0 & \text{otherwise}.\end{cases}$

5.4 Let $n\in\mathbb{N}$ and then

$$p(n\,|\,\{2,4\})=\left(\frac{n+2}{2}\right)\left(\frac{n+1}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)+\frac{1}{2}\left(\left\lfloor\frac{n}{4}\right\rfloor-\left\lfloor\frac{n-1}{2}\right\rfloor\right).$$

5.5 Let $n\in\mathbb{N}$ and then

$$p(n\,|\,\{2,4,6\})=\left\{\frac{(n+6)^2}{24}\left(\frac{n+1}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)\right\}.$$

5.6 Let $n\in\mathbb{N}$ and then

$$p(n\,|\,\{2,4,6,8\})=\left\{\left(\frac{(n+2)(n+14)^2+128}{576}\right)\left(\frac{n+1}{2}-\left\lfloor\frac{n+1}{2}\right\rfloor\right)\right.$$
$$\left.+\left(\frac{n+6}{32}\right)\left(\left\lfloor\frac{n}{4}\right\rfloor-\left\lfloor\frac{n-1}{4}\right\rfloor\right)\right\}.$$

6. Conflict of Interest

The author of this study conflicts that there is no conflict of interest with any financial organization regarding the material discussed in the manuscript.

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