

## The use of generating function and cyclotomic polynomials to discover and prove some new restricted integer partitions

Sonexay Songvilay<sup>1,\*</sup>, Thongsouk Saybounheang<sup>2</sup>, Innavanh Namkhaluck<sup>3</sup>, Pinkham Thanavanh<sup>4</sup>, Phuangmaly Siharath<sup>5</sup>, Parnthong Vongbouns<sup>6</sup> and Sonephanom Sengthavideth<sup>7</sup>

Research and Academic Service Division, Faculty of Natural Sciences, National University of Laos, Lao PDR

\*Correspondence: Sonexay

Songvilay, Research and Academic Service Division, Faculty of Natural Sciences, National University of Laos, PDR. Tel: +856 20 22220456

Email:

[songvilay.sonexay@gmail.com](mailto:songvilay.sonexay@gmail.com)

<sup>1,2,3,4,5,6</sup>Faculty of Natural Sciences, National University of Laos

<sup>7</sup>Department of Mathematics, Faculty of Natural Sciences, Champasack University

### Abstract

The purposes of this research were to discovered some new partition identities valid for all positive integers  $n$  of the general type  $p_1(n) = p_2(n)$ , where  $p_1(n)$  is the number of partitions of  $n$  and the parts of  $n$  are subject to a first restriction and  $p_2(n)$  is the number of partitions of  $n$  and the parts of  $n$  are subject to the second restriction, which is different from the first one. These partition identities are proved by using the generating function and properties of cyclotomic polynomials.

**Keywords:** *Restricted partitions, Generating function and Cyclotomic polynomials.*

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## 1. Introduction

A **partition** of the positive integers  $n$  is a way of writing  $n$  as a sum of positive integers in the form  $n = n_1 + n_2 + \dots + n_k$  where the order of the integers in the sum does not matter, and the integers  $n_1, n_2, \dots, n_k$  are called the **parts** of the partition of  $n$ . The number of partitions  $n$  is denoted by  $p(n)$ , and we define  $p(0) = 1$ . For example, we have  $p(8) = 22$  because there are 22 different partitions of 8 namely: 8, 7+1, 6+2, 6+1+1, 5+3, 5+2+1, 5+1+1+1, 4+4, 4+3+1, 4+2+2, 4+2+1+1, 4+1+1+1+1, 3+3+2, 3+3+1+1, 3+2+2+1, 3+2+1+1+1, 3+1+1+1+1+1, 2+2+2+2, 2+2+2+1+1, 2+2+1+1+1+1, 2+1+1+1+1+1+1 and 1+1+1+1+1+1+1+1.

A **restricted partition** is a partition of some kinds of restriction which is imposed upon the parts, and the number of partitions is often interesting to satisfy some condition in the form

$p(n | [\text{condition}])$ .

In 1979, Henry L. Alder used the generating function to find and prove partition identities concerning both unrestricted and restricted partitions. In particular, by modifying the Euler's theorem (1748), he proved that the number of partitions of  $n$ , **where each part appears at most twice** equals the number of partitions of  $n$  **into parts which are not divisible by 3**, and the number of partitions of  $n$ , **where each part appears at least twice** equals the number of partitions of  $n$  **into parts which are not congruent to  $\pm 1 \pmod{6}$** .

In this paper, the results of Alder (1979) will be extended. In particular, his technique is used to find and prove that the number of partitions of  $n$  in the first condition equals the number of partitions of  $n$  into parts with the second condition.

Therefore, the purpose of finding some new restricted integer partitions identities which are valid for all integers  $n$  of the general type  $p_1(n) = p_2(n)$  is interestingly studied by extending the results of Alder and these partition identities are proved by using the generating function and cyclotomic polynomials.

### 1. Materials and Methods

In this section, we use some definitions, theorems, and properties of the generating functions and cyclotomic polynomials to find and prove partition identities involving the restricted partitions.

**Definition 2.1**  $f(x)$  is said to be a generating function for the sequence  $\{a_0, a_1, a_2, \dots, a_n \dots\}$  if, when  $f(x)$  is writing as a power series, then

$$f(x) = a_0 + a_1x + a_2x^2 + \dots \dots$$

In considering sums as in Eq. (2.1), the authors do not have to concern with questions of convergence of the series since the coefficients are only the interesting once in this paper. A **formal power series** in  $x$  is considered. There exists a theorem of formal power series, of which an excellent account is given by **Ivan Niven**. The following results will be used thoroughly in the next section.

**Theorem 2.1** (Euler, 1748). The generating function for  $p(n)$  is given by

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1-x^j}$$

**Theorem 2.2** (Euler equality theorem, 1748). If  $n$  is a positive integer, then

$$p(n | \text{odd parts}) = p(n | \text{distinct parts}).$$

That is there are the same number of partitions of  $n$  into odd parts as there are partitions of  $n$  into distinct parts.

**Definition 2.2** (Erickson & Vazzana, 2008). The order  $n$  cyclotomic polynomial  $\Phi_n$  is the monic polynomial whose roots are the

distinct primitive  $n^{\text{th}}$  roots of unity. That is

$$\Phi_n(x) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} (x - \zeta_n^k), \text{ where } \zeta_n \text{ is a}$$

primitive  $n^{\text{th}}$  roots of unity.

Note that the degree of  $\Phi_n(x)$  is  $\phi(n)$ ,

where  $\phi$  is the Euler phi-function.

**Theorem 2.3** (Erickson & Vazzana, 2008). For  $n \geq 1$ ,

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

**Corollary 2.1** (Erickson & Vazzana, 2008). For  $p$  prime,

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \dots + x + 1.$$

### 2. Results

In this section, we start to discover and prove some new restricted integer partitions by using the generating function from the following problem 3.1.

**Problem 3.1** Is there an identity which states that the number of partitions of  $n$ , where **each part appears exactly twice** equals the number of partitions of  $n$  into parts taken from some fixed set of integers denote it by  $S$  and, if so, what is  $S$ ?

**Solution.** Let us assume that there exists such a set  $S$  and denote its elements by  $a_1, a_2, a_3, \dots$ , where  $a_1 < a_2 < a_3 < \dots$ . Then  $a_1, a_2, a_3, \dots$ , so are needed to determine that the generating function for the number partitions of  $n$ , where **each part appears exactly twice**, and it is denoted by  $f(x)$  which is equal to generating function for the number of partitions of  $n$  into parts taken from the set  $S$ , it is denoted again by  $s(x)$ . Now the generating function of  $f(x)$  is given by:

$$f(x) = (1+x^2)(1+x^4)(1+x^6)(1+x^8)(1+x^{10})\dots$$

and the generating function of  $s(x)$  is given by:

$$s(x) = (1+x^{a_1} + x^{2a_1} + \dots)(1+x^{a_2} + x^{2a_2} + \dots)(1+x^{a_3} + x^{2a_3} + \dots)\dots$$

$$= \frac{1}{(1-x^{a_1})(1-x^{a_2})(1-x^{a_3})\dots}$$

If  $f(x) = s(x)$ , then the term with lowest exponent of  $x$  in  $f(x)$  is  $x^2$ , must equal the term lowest exponent of  $x$  in  $s(x)$  is  $x^{a_1}$ , and

since  $a_1 < a_2 < a_3 < \dots$ , so that  $a_1 = 2$  is assumed. Multiplying both sides equation  $f(x) = s(x)$  by  $(1-x^2)$  and then the below result is obtained

$$(1+x^6)(1+x^{10})(1-x^{16})\dots = \frac{1}{(1-x^{a_2})(1-x^{a_3})\dots}$$

If  $(1-x^2)f(x) = (1-x^2)s(x)$ , then the term with lowest exponent of  $x$  in  $(1-x^2)f(x)$  is  $x^6$ , must equal the term lowest exponent of  $x$  in  $(1-x^2)s(x)$  is  $x^{a_2}$ , and since  $a_1 < a_2 < a_3 < \dots$ , so that  $a_2 = 6$  is assumed. Similarly,  $a_3 = 10, a_4 = 14, a_5 = 18, a_5 = 22, \dots$  is found. Thus,  $S = \{2, 6, 10, 14, 18, 22, \dots\}$ , that is,

the set of all numbers are congruent to  $2 \pmod{4}$ .

**Theorem 3.1** The number of partitions of  $n$ , where each part appears exactly twice equals the number of partitions of  $n$  into parts which are congruent to  $2 \pmod{4}$ .

**Proof.** Let  $f(x)$  be the generating function for the number of partitions of  $n$ , where each part appears exactly twice. Then

$$f(x) = \sum_{n=0}^{\infty} p(n \mid \text{each part appears exactly twice}) x^n = \prod_{i=1}^{\infty} (1+x^{2i}).$$

On the other hand, let  $s(x)$  be the generating function for the number of partitions of  $n$  into parts which are congruent to  $2 \pmod{4}$ . Then

$$s(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^{4i-2}} = \prod_{k \equiv 2 \pmod{4}} \frac{1}{1-x^k}.$$

And  $f(x) = s(x)$  must be showed.

From

$$\begin{aligned} f(x) &= \prod_{i=1}^{\infty} (1+x^{2i}) = \\ &= (1+x^2)(1+x^4)(1+x^6)(1+x^8)(1+x^{10})\dots \\ &= \frac{1-x^4}{1-x^2} \cdot \frac{1-x^8}{1-x^4} \cdot \frac{1-x^{12}}{1-x^6} \cdot \frac{1-x^{16}}{1-x^8} \cdot \frac{1-x^{20}}{1-x^{10}} \dots \\ &= \prod_{i=1}^{\infty} \frac{1}{1-x^{4i-2}} = \prod_{i \equiv 2 \pmod{4}} \frac{1}{1-x^i} = s(x). \end{aligned}$$

This completes the proof.

**Example 3.1** For  $n = 8$ , we have  $p(8) = 22$  and the partitions of 8 are: 8, 7+1, 6+2, 6+1+1, 5+3, 5+2+1, 5+1+1+1, 4+4, 4+3+1, 4+2+2, 4+2+1+1, 4+1+1+1+1, 3+3+2, 3+3+1+1, 3+2+2+1, 3+2+1+1+1, 3+1+1+1+1+1, 2+2+2+2, 2+2+2+1+1, 2+2+1+1+1+1, 2+1+1+1+1+1+1 and 1+1+1+1+1+1+1+1.

Then the partitions of 8, where each part appears exactly twice are: 4+4, 3+3+1+1. Thus, there are 2 such partitions. On the other hand, the partitions of 8 into parts which are congruent to  $2 \pmod{4}$  are: 6+2, 2+2+2+2. Thus, there are 2 such partitions.

Next, the Problem 1 can be replaced by using some condition to find the set  $S$  of integer as follows:

**Problem 3.2** Replace in Problem 3.1 “each part appears exactly twice” by “no part appears exactly two or three times”.

**Solution.** Let's assume that there exists such a set  $S$  and denote its elements by  $a_1, a_2, a_3, \dots$ , where  $a_1 < a_2 < a_3 < \dots$ . Then  $a_1, a_2, a_3, \dots$ , so are needed to determine that the generating function for the number partitions of  $n$ , where *no part appears exactly two or three times*, and denoted by  $f(x)$  which is equal to generating function for the number of partitions of  $n$  into parts taken from the set  $S$ , denoted again by  $s(x)$ . Now the generating function of  $f(x)$  is given by:

$$f(x) = (1 + x + x^4 + x^5 + x^6 + \dots)(1 + x^2 + x^8 + x^{10} + x^{12} + \dots) \cdot (1 + x^3 + x^{12} + x^{15} + x^{18} + \dots)(1 + x^4 + x^{16} + x^{20} + x^{24} + \dots) \dots$$

and the generating function of  $s(x)$  is given by:

$$s(x) = (1 + x^{a_1} + x^{2a_1} + \dots)(1 + x^{a_2} + x^{2a_2} + \dots)(1 + x^{a_3} + x^{2a_3} + \dots) \dots = \frac{1}{(1 - x^{a_1})(1 - x^{a_2})(1 - x^{a_3}) \dots}$$

If  $f(x) = s(x)$ , then the term with lowest exponent of  $x$  in  $f(x)$  is  $x$ , must equal the term lowest exponent of  $x$  in  $s(x)$  is  $x^{a_1}$ , and since  $a_1 < a_2 < a_3 < \dots$ , so are assumed that

$$(1 + x^6 + x^8)(1 + x^3 + x^{12} + x^{15} + x^{18} + \dots)(1 + x^4 + x^{16} + x^{20} + x^{24} + \dots) \dots = \frac{1}{(1 - x^{a_2})(1 - x^{a_3}) \dots}$$

If  $(1-x)f(x) = (1-x)s(x)$ , then the term with lowest exponent of  $x$  in  $(1-x)f(x)$  is  $x^3$ , and must equal the term lowest exponent of  $x$  in  $(1-x)s(x)$  is  $x^{a_2}$ , and since  $a_1 < a_2 < a_3 < \dots$ ,  $(1 + x^8 - x^{14} + x^{18} + x^{20})(1 + x^4 + x^{16} + x^{20} + x^{24} + \dots)$

$$\cdot (1 + x^5 + x^{20} + x^{25} + x^{30} + \dots) \dots = \frac{1}{(1 - x^{a_3})(1 - x^{a_4}) \dots}$$

If  $(1-x)(1-x^3)f(x) = (1-x)(1-x^3)s(x)$ , then the term with lowest exponent of  $x$  in  $(1-x)(1-x^3)f(x)$  is  $x^4$  and must equal the term lowest exponent of  $x$  in  $(1-x)(1-x^3)s(x)$  is  $x^{a_3}$ , and since  $a_1 < a_2 < a_3 < \dots$  so are assumed that  $a_3 = 4$ . Similarly,  $a_4 = 5, a_5 = 6, a_6 = 7, a_7 = 9, a_8 = 11, a_9 = 12, a_{10} = 13, \dots$  are found. Thus,  $S = \{1, 3, 4, 5, 6, 7, 9, 11, 13, \dots\}$ , that

$$f(x) = \sum_{n=0}^{\infty} p(n \mid \text{no part appears exactly two or three times}) x^n = \prod_{i=1}^{\infty} (1 + x + x^{4i} + x^{5i} + \dots)$$

On the other hand, let  $s(x)$  be the generating function for the number of partitions of  $n$  into parts which are not congruent to  $\pm 2 \pmod{12}$ . Then

$$s(x) = \prod_{i \not\equiv \pm 2 \pmod{12}} \frac{1}{1 - x^i}$$

And  $f(x) = s(x)$  must showed that.

$$\text{From } f(x) = \prod_{i=1}^{\infty} (1 + x + x^{4i} + x^{5i} + \dots) = \prod_{i=1}^{\infty} (1 + x + x^{2i} + x^{3i} + x^{4i} + x^{5i} + \dots - x^{2i} - x^{3i})$$

$a_1 = 1$ . Multiplying both sides equation  $f(x) = s(x)$  by  $(1-x)$ , the below result is obtained

so are assumed that  $a_2 = 3$ . Multiplying both sides equation  $(1-x)f(x) = (1-x)s(x)$  by  $(1-x^3)$ , the below result is obtained

is, the set of all numbers are not congruent to  $\pm 2 \pmod{12}$ .

**Theorem 3.2** The number of partitions of  $n$ , where no part appears exactly two or three times equals the number of partitions of  $n$  into parts which are not congruent to  $\pm 2 \pmod{12}$ .

**Proof.** Let  $f(x)$  be the generating function for the number of partitions of  $n$ , where no part appears exactly two or three times. Then

$$\begin{aligned} &= \prod_{i=1}^{\infty} \left( \frac{1}{1-x^i} - x^{2i} - x^{3i} \right) = \prod_{i=1}^{\infty} \left( \frac{1-x^{2i}+x^{4i}}{1-x^i} \right) = \prod_{i=1}^{\infty} \frac{1+x^{6i}}{(1-x^i)(1+x^{2i})} \\ &= \prod_{i=1}^{\infty} \frac{(1-x^{2i})(1-x^{12i})}{(1-x^i)(1-x^{4i})(1-x^{6i})} = \prod_{i=1}^{\infty} \frac{1}{(1-x^{2i-1})(1-x^{4i})(1-x^{6(2i-1)})} \\ &= \prod_{i \neq \pm 2 \pmod{12}} \frac{1}{1-x^i} = s(x). \end{aligned}$$

This completes the proof.

**Example 3.2** From example 4.1 and  $p(8) = 22$ . Then the partitions of 8, where no part appears exactly two or three times are: 8, 7+1, 6+2, 5+3, 5+2+1, 4+3+1, 4+1+1+1+1, 3+1+1+1+1+1, 2+2+2+2, 2+1+1+1+1+1+1 and 1+1+1+1+1+1+1+1. Thus, there are 11 such partitions. On the other hand, the partitions of 8 into parts which are not congruent to  $\pm 2 \pmod{12}$  are: 8, 7+1, 6+1+1, 5+3, 5+1+1+1, 4+4, 4+3+1, 4+1+1+1+1, 3+3+1+1, 3+1+1+1+1+1 and 1+1+1+1+1+1+1+1. Thus, there are 11 such partitions.

Next, we present some new restricted partition identities using the cyclotomic polynomials are represented in this paper. The  $n^{\text{th}}$  cyclotomic polynomial  $\Phi_n(x)$  is defined to be the polynomial whose roots are the primitive  $n^{\text{th}}$  root of unity:

$$\Phi_n(x) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} (x - \zeta_n^k).$$

Recall that  $\Phi_1(x) = x - 1$  and  $\Phi_p(x) = 1 + x + x^2 + \dots + x^{p-1}$ , where  $p$  is a prime

$$\Phi_{10}(x) = \frac{x^{10} - 1}{\Phi_1(x)\Phi_2(x)\Phi_5(x)} = \frac{x^{15} - 1}{(x-1)(x+1)(x^4 + x^3 + x^2 + x + 1)}.$$

Let  $f(x)$  be the generating function for the number of partitions of  $n$ , where no part appears exactly one or three times. Then

$$f(x) = \sum_{n=0}^{\infty} p(n | \text{no part appears exactly one or three times}) x^n = \prod_{i=1}^{\infty} (1 + x^{2i} + x^{4i} + x^{5i} + \dots).$$

On the other hand, let  $s(x)$  be the generating function for the number of partitions of  $n$  into parts which are congruent to  $0, \pm 2, \pm 4, 5 \pmod{10}$ . Then

$$s(x) = \prod_{i=0, \pm 2, \pm 4, 5 \pmod{10}} \frac{1}{1-x^i}.$$

And  $f(x) = s(x)$  must be showed.

number. The cyclotomic polynomial  $\Phi_n(x)$  can be computed by dividing  $x^n - 1$  by the cyclotomic polynomials of the proper divisor of  $n$  previously compute recursively by the same method:

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{\substack{d|n \\ d < n}} \Phi_d(x)}.$$

The cyclotomic polynomials

$$\Phi_{10}(x) = 1 - x + x^2 - x^3 + x^4,$$

$$\Phi_{14}(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 \text{ and}$$

$\Phi_{15}(x) = 1 - x + x^3 - x^4 + x^5 - x^7 + x^8$  is used in the proofs of Theorems 3.3, 3.4 and 3.5 respectively.

**Theorem 3.3** The number of partitions of  $n$ , where no part appears exactly one or three times equals the number of partitions of  $n$  into parts which are congruent to  $0, \pm 2, \pm 4, 5 \pmod{10}$ .

**Proof.** The key idea of this proof is to use cyclotomic polynomial  $\Phi_{10}(x) = 1 - x + x^2 - x^3 + x^4$ . First note that:

$$\begin{aligned}
 \text{From } f(x) &= \prod_{i=1}^{\infty} (1 + x^{2i} + x^{4i} + x^{5i} + \dots) = \prod_{i=1}^{\infty} (1 + x + x^{2i} + x^{3i} + x^{4i} + x^{5i} + \dots - x^i - x^{3i}) \\
 &= \prod_{i=1}^{\infty} \left( \frac{1}{1-x^i} - x^i - x^{3i} \right) = \prod_{i=1}^{\infty} \left( \frac{1-x^i + x^{2i} - x^{3i} + x^{4i}}{1-x^i} \right) \\
 &= \prod_{i=1}^{\infty} \frac{\Phi_{10}(x)}{1-x^i}, \text{ where } \Phi_{10}(x) = \frac{x^{10} - 1}{\Phi_1(x)\Phi_2(x)\Phi_5(x)} \\
 &= \prod_{i=1}^{\infty} \frac{x^{10i} - 1}{(1-x^i)(x^i - 1)(x^i + 1)(x^{4i} + x^{3i} + x^{2i} + x^i + 1)} \\
 &= \prod_{i=1}^{\infty} \frac{(x^{5i} - 1)(x^{5i} + 1)}{(x^{2i} - 1)(1 - x^{5i})} = \prod_{i=1}^{\infty} \frac{1 + x^{5i}}{1 - x^{2i}} = \prod_{i=1}^{\infty} \frac{1 - x^{10i}}{(1 - x^{2i})(1 - x^{5i})} \\
 &= \prod_{i=1}^{\infty} \frac{1}{(1 - x^{2i})(1 - x^{5(2i-1)})} = \prod_{i=0, \pm 2, \pm 4, 5(\text{mod } 10)} \frac{1}{1 - x^i} = s(x).
 \end{aligned}$$

This completes the proof.

**Example 3.3** From example 4.1 and  $p(8) = 22$  then the partitions of 8, where no part appears exactly one or three times are: 4+4, 3+3+1+1, 2+2+2+2, 2+2+1+1+1+1 and 1+1+1+1+1+1+1+1. Thus, there are 5 such partitions. On the other hand, the partitions of 8 into parts which are congruent to  $0, \pm 2, \pm 4, 5(\text{mod } 10)$  are: 8, 6+2, 4+4, 4+2+2 and 2+2+2+2. Thus, there are 5 such partitions.

**Theorem 3.4** The number of partitions of  $n$ , where no part appears exactly one or three or five times equals the number of partitions of  $n$  into parts which are congruent to  $0, \pm 2, \pm 4, \pm 6, 7(\text{mod } 14)$ .

**Proof.** Let  $f(x)$  be the generating function for the number of partitions of  $n$ , where no part appears exactly one or three or five times. Then

$$\begin{aligned}
 f(x) &= \sum_{n=0}^{\infty} p(n \mid \text{no part appears exactly one or three or five times}) x^n \\
 &= \prod_{i=1}^{\infty} (1 + x^{2i} + x^{4i} + x^{6i} + x^{7i} + \dots).
 \end{aligned}$$

On the other hand, let  $s(x)$  be the generating function for the number of partitions of  $n$  into parts which are congruent to  $0, \pm 2, \pm 4, \pm 6, 7(\text{mod } 14)$ . Then

$$s(x) = \prod_{i=0, \pm 2, \pm 4, \pm 6, 7(\text{mod } 14)} \frac{1}{1 - x^i}.$$

And  $f(x) = s(x)$  must be showed that.

$$\begin{aligned}
 \text{From } f(x) &= \prod_{i=1}^{\infty} (1 + x^{2i} + x^{4i} + x^{6i} + x^{7i} + \dots) \\
 &= \prod_{i=1}^{\infty} (1 + x + x^{2i} + x^{3i} + x^{4i} + x^{5i} + \dots - x^i - x^{3i} - x^{5i}) \\
 &= \prod_{i=1}^{\infty} \left( \frac{1}{1-x^i} - x^i - x^{3i} - x^{5i} \right) = \prod_{i=1}^{\infty} \left( \frac{1-x^i + x^{2i} - x^{3i} + x^{4i} - x^{5i} + x^{6i}}{1-x^i} \right) \\
 &= \prod_{i=1}^{\infty} \frac{\Phi_{14}(x)}{1-x^i}, \text{ where } \Phi_{14}(x) = \frac{x^{14} - 1}{\Phi_1(x)\Phi_2(x)\Phi_7(x)} \\
 &= \prod_{i=1}^{\infty} \frac{x^{14i} - 1}{(1-x^i)(x^i - 1)(x^i + 1)(x^{6i} + x^{5i} + x^{4i} + x^{3i} + x^{2i} + x^i + 1)}
 \end{aligned}$$

$$\begin{aligned}
 &= \prod_{i=1}^{\infty} \frac{(x^{7i} - 1)(x^{7i} + 1)}{(x^{2i} - 1)(1 - x^{7i})} = \prod_{i=1}^{\infty} \frac{1 + x^{7i}}{1 - x^{2i}} = \prod_{i=1}^{\infty} \frac{1 - x^{14i}}{(1 - x^{2i})(1 - x^{7i})} \\
 &= \prod_{i=1}^{\infty} \frac{1}{(1 - x^{2i})(1 - x^{7(2i-1)})} = \prod_{i=0, \pm 2, \pm 4, \pm 6, 7 \pmod{14}} \frac{1}{1 - x^i} = s(x).
 \end{aligned}$$

This completes the proof.

**Proof.** Let  $f(x)$  be the generating

**Theorem 3.5** The number of partitions of  $n$ , where no part appears exactly one or two or four or seven times equals the number of partitions of  $n$  into parts which are congruent to  $0, \pm 3, \pm 5, \pm 6 \pmod{15}$ .

function for the number of partitions of  $n$ , where no part appears exactly one or two or four or seven times. Then

$$\begin{aligned}
 f(x) &= \sum_{n=0}^{\infty} p(n \mid \text{no part appears exactly one or two or four or seven times}) x^n \\
 &= \prod_{i=1}^{\infty} (1 + x^{3i} + x^{5i} + x^{6i} + x^{8i} + x^{9i} + \dots).
 \end{aligned}$$

On the other hand, let  $s(x)$  be the generating function for the number of partitions of  $n$  into parts which are congruent to  $0, \pm 3, \pm 5, \pm 6 \pmod{15}$ . Then

$$s(x) = \prod_{i=0, \pm 3, \pm 5, \pm 6 \pmod{15}} \frac{1}{1 - x^i}.$$

And  $f(x) = s(x)$  must be showed that.

From  $f(x) = \prod_{i=1}^{\infty} (1 + x^{3i} + x^{5i} + x^{6i} + x^{8i} + x^{9i} + \dots)$ .

$$\begin{aligned}
 &= \prod_{i=1}^{\infty} (1 + x + x^{2i} + x^{3i} + x^{4i} + x^{5i} + \dots - x^i - x^{2i} - x^{4i} - x^{7i}) \\
 &= \prod_{i=1}^{\infty} \left( \frac{1}{1 - x^i} - x^i - x^{2i} - x^{4i} - x^{7i} \right) \\
 &= \prod_{i=1}^{\infty} \frac{1 - x^i + x^{3i} - x^{4i} + x^{5i} - x^{7i} + x^{8i}}{1 - x^i} \\
 &= \prod_{i=1}^{\infty} \frac{\Phi_{15}(x)}{1 - x^i}, \text{ where } \Phi_{15}(x) = \frac{x^{15} - 1}{\Phi_1(x)\Phi_3(x)\Phi_5(x)} \\
 &= \prod_{i=1}^{\infty} \frac{x^{15i} - 1}{(1 - x^i)(x^i - 1)(x^{2i} + x^i + 1)(x^{4i} + x^{3i} + x^{2i} + x^i + 1)} \\
 &= \prod_{i=1}^{\infty} \frac{1 - x^{15i}}{(1 - x^{3i})(1 - x^{5i})} = \prod_{i=1}^{\infty} \frac{1}{(1 - x^{3i})(1 - x^{5(3i-2)})(1 - x^{5(3i-1)})} \\
 &= \prod_{i=0, \pm 3, \pm 5, \pm 6 \pmod{15}} \frac{1}{1 - x^i} = s(x).
 \end{aligned}$$

This completes the proof.

### 3. Discussion

**3.1** The results of this research are close to the research of Henry L. Alder (1969), who studied **Partition Identities-From Euler to the Present** and obtained *the number of partitions*

*of  $n$  into parts differing by at least 2 is equal to the partitions of  $n$  into parts which are congruent 1 or 4 (mod 5).* In this research, a similar result found in this research is to use the generating function to find and prove partition identities, however, the main results of this

research are difference such as *the number of partitions of  $n$ , where no part appears exactly two or three times equals the number of partitions of  $n$  into parts which are not congruent to 2 or 10 (mod 12)*.

**3.2** The results of this research are close to the research of Henry L. Alder (1979), who studied **the use of generating function to discover and prove partition identities** and obtained *the number of partitions of  $n$ , where each part appears at most twice and is for all  $n$  equal to the number of partitions of  $n$  into parts which are not divisible by 3*. In this research, a similar result found in this research is to use the generating function to find and prove partition identities, however, the main results of this research are difference such as *the number of partitions of  $n$ , where each part appears exactly twice equals the number of partitions of  $n$  into parts which are congruent to 2(mod 4)*.

**3.3** The results of this research are close to the research of G. E. Andrews and E. Kimmo (2004), who studied **Integer Partitions** which the extension research of Subbarao (1971b) and obtained *the number of partitions of  $n$  in which each part appears exactly 2, 3 or 5 times equals the number of partitions of  $n$  into parts congruent to  $\pm 2, \pm 3, 6$ (mod 12)*. In this research, a similar result found in this research is to use the generating function to find and prove partition identities, however, the main results of this research are difference because some relevant theories such as the generating

$$p(n | \text{each part appears exactly twice}) = p(n | \text{parts which are congruent to } 2(\text{mod } 4)).$$

**4.2** The number of partitions of  $n$ , where no part appears exactly two or three times equals the number of partitions of  $n$  into parts which are not congruent to  $\pm 2(\text{mod } 12)$ .

$$p(n | \text{no part appears exactly two or three times}) =$$

$$p(n | \text{parts which are not congruent to } \pm 2(\text{mod } 12)).$$

**4.3** The number of partitions of  $n$ , where no part appears exactly one or three times equals the number of partitions of  $n$  into parts which are congruent to  $0, \pm 2, \pm 4, 5$  (mod 10).

$$p(n | \text{no part appears exactly one or three times}) =$$

$$p(n | \text{parts which are congruent to } 0, \pm 2, \pm 4, 5(\text{mod } 10)).$$

**4.4** The number of partitions of  $n$ , where no part appears exactly one or three or five times equals the number of partitions of  $n$  into parts which are congruent to  $0, \pm 2, \pm 4, \pm 6, 7$ (mod 14).

$$p(n | \text{no part appears exactly one or three or five times}) =$$

function, properties of cyclotomic polynomial and congruent are used to find and prove partition identities and obtained with the following results:

- *The number of partitions of  $n$ , where no part appears exactly one or three times equals the number of partitions of  $n$  into parts which are congruent to  $0, \pm 2, \pm 4, 5$  (mod 10).*

- *The number of partitions of  $n$ , where no part appears exactly one or three or five times equals the number of partitions of  $n$  into parts which are congruent to  $0, \pm 2, \pm 4, \pm 6, 7$  (mod 14).*

- *The number of partitions of  $n$ , where no part appears exactly one or two or four or seven times equals the number of partitions of  $n$  into parts which are congruent to  $0, \pm 3, \pm 5, \pm 6$  (mod 15).*

#### 4. Conclusion

The main objective of this study of some new partition identities concerning restricted partition valid for all positive integers  $n$  of the general type:

$$p_1(n) = p_2(n) \text{ is studied,}$$

where the two conditions are different by using the generating functions and cyclotomic polynomials to discover and prove the following results:

**4.1** The number of partitions of  $n$ , where each part appears exactly twice equals the number of partitions of  $n$  into parts which are congruent to  $2(\text{mod } 4)$

$p(n | \text{parts which are congruent to } 0, \pm 2, \pm 4, \pm 6, 7 \pmod{14})$ .

**4.5** The number of partitions of  $n$ , where no part appears exactly one or two or four or seven times equals the number of partitions of  $n$  into parts which are congruent to  $0, \pm 3, \pm 5, \pm 6 \pmod{15}$ .

$p(n | \text{no part appears exactly one or two or four or seven times}) =$

$p(n | \text{parts which are congruent to } 0, \pm 3, \pm 5, \pm 6 \pmod{15})$ .

**5. Conflict of Interest**

The author of this study conflicts that there is no conflict of interest with any financial organization regarding the material discussed in the manuscript.

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